# Formulas for Calculating Zoo and Zoe of Coupled Microstrip

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This page calculates Zoo and Zoe for coupled microstrip. It uses the equation in "Accurate Wide-Range Design Equations for the Frequency Dependent Characteristics...", Kirschning, Jansen, IEEE MITT Jan 1984, Page 83 onwards.

The Effective Line Permittivity is from "Handbook of Microwave and Optical Components", Chang, Chapter 1, Table 1.16. This has a correction for line thickness.

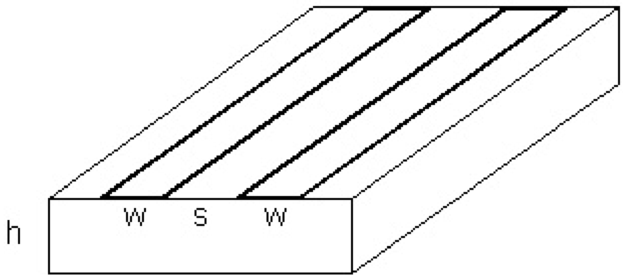
No calculation has been made for frequency as the dimensions are so small that dispersion will not cause a problem, Refer to sheet for Zo. The results are compared to ADS linecalc and are pretty close but up to 10% out for thick lines; thin lines are ok.

**Definitions:**

$$\mu m := m \cdot 10^{-6}$$

$$c := 3 \times 10^{8} \cdot \frac{m}{s}$$

$$u := \frac{w}{h} \qquad g := \frac{s}{h} \qquad p := \frac{t}{h}$$




**Width Correction**

$$u_{eff}(u,p) := \begin{cases} u + 1.25 \cdot \left(\frac{p}{\pi}\right) \cdot \left(1 + \ln\left(\frac{4 \cdot \pi \cdot u}{p}\right)\right) & \text{if } u < \frac{1}{2 \cdot \pi} \\ u + 1.25 \cdot \left(\frac{p}{\pi}\right) \cdot \left(1 + \ln\left(\frac{4 \cdot \pi \cdot u}{p}\right)\right) & \text{if } u = \frac{1}{2 \cdot \pi} \\ u + 1.25 \cdot \left(\frac{p}{\pi}\right) \cdot \left(1 + \ln\left(\frac{2}{p}\right)\right) & \text{if } u > \frac{1}{2 \cdot \pi} \end{cases}$$

**Single Microstrip Line Effective Permittivity at dc**

$$\varepsilon_{eff.0}(\varepsilon_r, u, p) := \begin{cases} \frac{(\varepsilon_r + 1)}{2} + \frac{(\varepsilon_r - 1)}{2} \cdot \left[\left(1 + \frac{12}{u}\right)^{-0.5} + 0.04 \cdot \left(1 - \frac{1}{u}\right)^2\right] - \frac{(\varepsilon_r - 1)}{4.6} \cdot p \cdot \frac{1}{\sqrt{u}} & \text{if } u < 1 \\ \frac{(\varepsilon_r + 1)}{2} + \frac{(\varepsilon_r - 1)}{2} \cdot \left[\left(1 + \frac{12}{u}\right)^{-0.5} + 0.04 \cdot \left(1 - \frac{1}{u}\right)^2\right] - \frac{(\varepsilon_r - 1)}{4.6} \cdot p \cdot \frac{1}{\sqrt{u}} & \text{if } u = 1 \\ \frac{(\varepsilon_r + 1)}{2} + \frac{(\varepsilon_r - 1)}{2} \left[\left(1 + \frac{12}{u}\right)^{-0.5}\right] - \frac{(\varepsilon_r - 1)}{4.6} \cdot p \cdot \frac{1}{\sqrt{u}} & \text{if } u > 1 \end{cases}$$

$$\varepsilon_{eff.0}\left(4.1, \frac{7}{7}, \frac{0.1}{7}\right) = 2.97$$

**Even Mode Effective Permittivity at dc**

$$v(u,g) := u \cdot \frac{\left(20 + g^2\right)}{\left(10 + g^2\right)} + g \cdot \exp(-g)$$

$$b_e(\varepsilon_r) := 0.564 \cdot \left(\frac{\varepsilon_r - 0.9}{\varepsilon_r + 3.0}\right)^{0.053}$$

$$a_e(u,g) := 1 + \frac{1}{49} \cdot \ln\left[\frac{v(u,g)^4 + \left(\frac{v(u,g)}{52}\right)^2}{v(u,g)^4 + 0.432}\right] + \frac{1}{18.7} \cdot \ln\left[1 + \left(\frac{v(u,g)}{18.1}\right)^3\right]$$

$$\varepsilon_{eff.e.0}(\varepsilon_r, u, g) := \frac{\varepsilon_r + 1}{2} + \frac{\varepsilon_r - 1}{2}\cdot\left(1 + \frac{10}{v(u,g)}\right)^{-a_e(u,g)\cdot b_e(\varepsilon_r)}$$

This agrees closely with HPADS for all values

$$\varepsilon_{eff.e.0}\left(4.1, \frac{5}{7}, \frac{10}{7}\right) = 3.089$$

### Impedance Zo

$$Zo(\varepsilon_r, u, p) := \begin{cases} \frac{60}{\sqrt{\varepsilon_{eff.e.0}(\varepsilon_r, u_{eff}(u,p), p)}}\cdot \ln\left(\frac{8}{u_{eff}(u,p)} + 0.25\cdot u_{eff}(u,p)\right)\cdot ohm & \text{if } u_{eff}(u,p) < 1 \\ \frac{60}{\sqrt{\varepsilon_{eff.e.0}(\varepsilon_r, u_{eff}(u,p), p)}}\cdot \ln\left(\frac{8}{u_{eff}(u,p)} + 0.25\cdot u_{eff}(u,p)\right)\cdot ohm & \text{if } u_{eff}(u,p) = 1 \\ \frac{120\cdot\pi}{\sqrt{\varepsilon_{eff.e.0}(\varepsilon_r, u_{eff}(u,p), p)}}\cdot \frac{1}{(u_{eff}(u,p) + 1.393 + 0.667\cdot\ln(1.444 + u_{eff}(u,p)))}\cdot ohm & \text{if } u_{eff}(u,p) > 1 \end{cases}$$

Compared to ADS this is 4% out for this narrow, thick tracks

$$Zo\left(4.1, \frac{5}{7}, \frac{2}{7}\right) = 69.314\,ohm$$

### Odd Mode Permittivity

$$b_o(\varepsilon_r) := \frac{0.747\cdot\varepsilon_r}{0.15 + \varepsilon_r} \qquad c_o(\varepsilon_r, u) := b_o(\varepsilon_r) - (b_o(\varepsilon_r) - 0.207)\cdot\exp(-0.414\cdot u) \qquad d_o(u) := 0.593 + 0.694\cdot\exp(-0.562\cdot u)$$

$$a_o(\varepsilon_r, u, p) := 0.7287\cdot\left(\varepsilon_{eff.0}(\varepsilon_r, u, p) - \frac{\varepsilon_r + 1}{2}\right)\cdot(1 - \exp(-0.179\cdot u))$$

$$\varepsilon_{eff.o.0}(\varepsilon_r, u, g, p) := \left[\left(\frac{\varepsilon_r + 1}{2} + a_o(\varepsilon_r, u, p)\right) - \varepsilon_{eff.0}(\varepsilon_r, u, p)\right]\cdot\exp\left(-c_o(\varepsilon_r, u)\cdot g^{d_o(u)}\right) + \varepsilon_{eff.0}(\varepsilon_r, u, p)$$

Compared to ADS this is 22% out for this narrow, thick track. Wider & reduced height tracks are in excellent agreement

$$\varepsilon_{eff.o.0}\left(4.1, \frac{5}{7}, \frac{4}{7}, \frac{2}{7}\right) = 2.586$$

### Even Mode Characteristic Impedance

$$Q_1(u) := 0.8695\cdot u^{0.194} \qquad Q_2(g) := 1 + 0.7519\cdot g + 0.189\cdot g^{2.31}$$

$$Q_3(g) := 0.1975 + \left[16.6 + \left(\frac{8.4}{g}\right)^6\right]^{-0.387} + \frac{1}{241}\cdot\ln\left[\frac{g^{10}}{1 + \left(\frac{g}{3.4}\right)^{10}}\right]$$

$$Q_4(u, g) := 2\cdot\frac{Q_1(u)}{Q_2(g)}\cdot\left[u^{-Q_3(g)}\cdot\left[\exp(-g)\cdot u^{-Q_3(g)} + (2 - \exp(-g))\right]\right]^{-1}$$

$$Z_{Le}(\varepsilon_r, u, g, p) := Zo(\varepsilon_r, u, p)\cdot\sqrt{\left(\frac{\varepsilon_{eff.0}(\varepsilon_r, u, p)}{\varepsilon_{eff.e.0}(\varepsilon_r, u, g)}\right)}\cdot\frac{1}{\left(1 - \frac{\sqrt{\varepsilon_{eff.0}(\varepsilon_r, u, p)}\cdot Zo(\varepsilon_r, u, p)\cdot Q_4(u, g)}{377ohm}\right)}$$

### Odd Mode Characteristic Impedance

$$Q_5(g) := 1.794 + 1.14 \cdot \ln\left(1 + \frac{0.638}{g + 0.517 \cdot g^{2.43}}\right)$$

$$Q_6(g) := 0.2305 + \frac{1}{281.3} \cdot \ln\left[\frac{g^{10}}{1 + \left(\frac{g}{5.8}\right)^{10}}\right] + \frac{\ln\left(1 + 0.598 \cdot g^{1.154}\right)}{5.1} \qquad Q_7(g) := \frac{10 + 190 \cdot g^2}{1 + 82.3 \cdot g^3}$$

$$Q_8(g) := \exp\left[-6.5 - 0.95 \cdot \ln(g) - \left(\frac{g}{0.15}\right)^5\right] \qquad Q_9(g) := \left(Q_8(g) + \frac{1}{16.5}\right) \cdot \ln\left(Q_7(g)\right)$$

$$Q_{10}(u, g) := \frac{Q_2(g) \cdot Q_4(u, g) - Q_5(g) \cdot \exp\left(\ln(u) \cdot Q_6(g) \cdot u^{-Q_9(g)}\right)}{Q_2(g)}$$

$$Z_{Lo}(\varepsilon_r, u, g, p) := Zo(\varepsilon_r, u, p) \cdot \sqrt{\left(\frac{\varepsilon_{eff.0}(\varepsilon_r, u, p)}{\varepsilon_{eff.o.0}(\varepsilon_r, u, g, p)}\right)} \cdot \frac{1}{\left(1 - \frac{\sqrt{\varepsilon_{eff.0}(\varepsilon_r, u, p)} \cdot Zo(\varepsilon_r, u, p) \cdot Q_{10}(u, g)}{377 \mathrm{ohm}}\right)}$$

**This table shows values calculated by this MathCAD sheet and values calculated by ADS LineCalc**

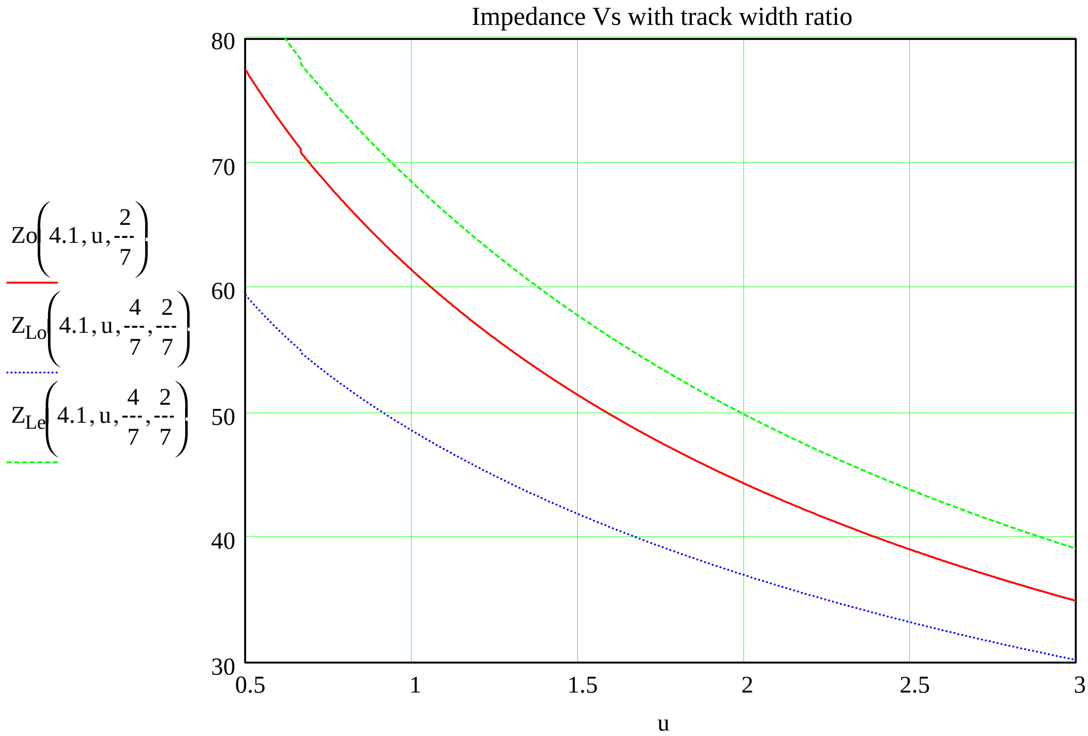
| w (um) | s(um) | h(um) | t(um) | Er | Zo | Zlo | Zle | ADS Zo | ADS Zlo | ADS Zle |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 4 | 7 | 2 | 4.1 | 69.31 | 53.78 | 76.43 | 66.77 | 47.55 | 93.75 |
| 10 | 4 | 7 | 2 | 4.1 | 52.63 | 42.71 | 59.13 | 50.99 | 38.73 | 67.13 |
| 15 | 4 | 7 | 2 | 4.1 | 42.66 | 35.81 | 47.98 | 41.63 | 32.95 | 52.60 |
| 20 | 4 | 7 | 2 | 4.1 | 35.99 | 30.97 | 40.34 | 35.30 | 28.77 | 43.34 |
| 5 | 10 | 7 | 2 | 4.1 | 69.31 | 62.49 | 71.21 | 71.52 | 61.31 | 83.84 |
| 10 | 10 | 7 | 2 | 4.1 | 52.63 | 48.17 | 55.01 | 54.00 | 47.48 | 61.41 |
| 15 | 10 | 7 | 2 | 4.1 | 42.66 | 39.49 | 44.78 | 43.76 | 39.11 | 48.96 |
| 20 | 10 | 7 | 2 | 4.1 | 35.99 | 33.59 | 37.79 | 36.92 | 33.30 | 40.81 |
| 5 | 10 | 7 | 0.1 | 4.1 | 81.77 | 73.38 | 89.31 | 83.33 | 74.82 | 92.81 |
| 10 | 10 | 7 | 0.1 | 4.1 | 58.67 | 53.44 | 63.91 | 60.00 | 54.50 | 66.08 |
| 15 | 10 | 7 | 0.1 | 4.1 | 46.42 | 42.79 | 50.32 | 47.51 | 43.55 | 51.82 |
| 20 | 10 | 7 | 0.1 | 4.1 | 38.57 | 35.87 | 41.58 | 39.55 | 36.49 | 42.76 |

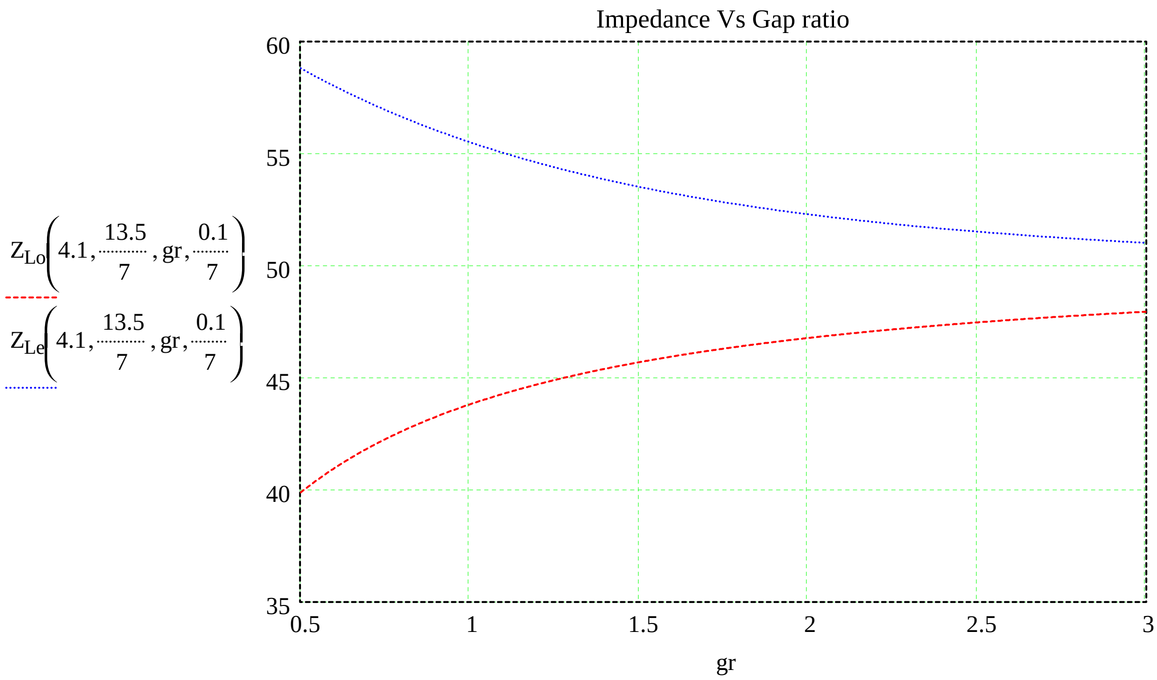
$$Zo\left(4.1, \frac{5}{7}, \frac{2}{7}\right) = 69.314 \mathrm{ohm}$$

$$Z_{Lo}\left(4.1, \frac{5}{7}, \frac{4}{7}, \frac{2}{7}\right) = 53.78 \mathrm{ohm}$$

$$Z_{Le}\left(4.1, \frac{5}{7}, \frac{4}{7}, \frac{2}{7}\right) = 76.432 \mathrm{ohm}$$

Probably the biggest weakness is Zo does not take into account the spacing. These equations are close but must have an error. However this does to allow the trends to be plotted


Impedance Vs with track width ratio
Zo(4.1, u, 2/7)
ZLo(4.1, u, 4/7, 2/7)
ZLe(4.1, u, 4/7, 2/7)
80
70
60
50
40
30
0.5
1
1.5
2
2.5
3
u


Impedance Vs Gap ratio
ZLo(4.1, 13.5/7, gr, 0.1/7)
ZLe(4.1, 13.5/7, gr, 0.1/7)
60
55
50
45
40
35
0.5
1
1.5
2
2.5
3
gr